**Linear Second Order Equations**

Consider the general linear form:



wherever both p and q are continous, there exists two linearly independent solutions. The two solutions are linearly independent where their Wronskian is non-zero . The Wronskian is defined as:



If the Wronskian is identically zero at all points, then the two functions are linearly dependent. If the functions are solutions of the same differential equation, and their Wronskian is zero anywhere, then it is zero everywhere, and they are linearly dependent.

Note we can explicitly determine W(x) when r(x) (it’s useful to have an equation for what we’ll do below). This equation for W(x) can be demonstrated as below. Multiply the equation for y1 by y2 and multiply the equation for y2 by y1 and we have:



And subtracting these two we have,



So we have the equation



which has the solution,



finally, for future reference, let's note that:



**Reduction of Order**

If one solution, y1(x), is found another, y2(x), can be found by substituting in y2(x) = v(x)y1(x). For consider let y1 be a solution and then plug y2 = vy1 into the equation:



And now we can solve this equation...



Now let's observe that:



So we can write:



and we may say that the second solution is given by:



**Example: Legendre’s Equation**

Let's consider Legendre's equation as an example.



has a solution y1 = x. A second solution would be:



which is the relevant legendre function of the second kind, as expected.

**Elimination of 1st Derivative**

Can always make a substitution y = vψ and choose v to eliminate the 1st derivative.



So if we choose:



And integrating,



we’ll have the equation:



Along another vein, such a second order differential equation can always be put in standard form with the substitution y = v\*ψ, where the first derivative is eliminated. Since this would put the differential equation in Schrodinger equation like form, and since the weight function of ψ would then be 1 , it must be the case that the weight function of y is w = 1/v2. So this means that the substitution v is simply 1/√w, where w is the weight function.

**Solution Methods for Constant Coefficient Equations**  
This looks like something of the sort:



where p and q are *constants*. General idea is to find any particular solution, yp, to the inhomogeneous ODE. Then find the general solution to the homogeneous equation, yh, and the sum, y = yh + yp will be the general solution to the entire equation. This is often used in PDE's for instance, when you have say a sphere with some potential specified on it, and some charge inside of it, or outside of it. You use the particular solution as the familiar integral over ρ/r2, or equivalently, you use the method of Green's function - same thing. Then you add to that particular solution the solution to the homogeneous equation - i.e. spherical harmonics series, and then determine the coefficients by matching the boundary conditions.

The simplest way to get yp is to choose a form that mimics r(x).

|  |  |
| --- | --- |
| **r(x)** | **yp(x)** |
| eax | Aeax |
| Acos(ωx) + Bsin(ωx) | Ccos(ωx) + Dsin(ωx) |
| Nth degree polynomial | Nth degree polynomial with variable coefs |
| Sum of above | Sum of above |
| Product of above | Product of above |

note: if any of trial yp's contain in them, either as sum or product, a solution to the homogenous equation, multiply the offending part by *x*, kind of like was done for the homogeneous solution in case of double root.

**Solution Methods for Non-constant Coefficient Inhomogeneous Equations**

If we have non-constant coefficient equations, like the one at the top basically,



or if we just have a particulary obstinate r(x), then we can avail ourselves of the method of **variation of parameters**. Again we split our solution into y = yh + yp. To get yp, we assume a particular solution of this form:



where y1,2 are homogeneous solutions. Then we substitute this into the ODE and require that u´1y1 + u´2y2 = 0 - the natural term to equate to zero when you take the first derivative of yp, this is just one half of the product rule. Then we end up with this equation:



These two simultaneous equations we can solve for u1(x) and u2(x)! We can use Cramer’s rule to get:



Integrating, we find:



The procedure can be naturally generalized to 3rd and higher order ODE’s.